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RESEARCH NOTE
A note on solutions to the <Q,r> inventory model for gamma lead-time demand

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Abstract *Namit and Chen recently created two algorithms to solve the <Q,r> inventory model for gamma lead-time demand without using tabulated values. However, other less complicated solutions that do not require the use of tabulated values are currently available. This note demonstrated the relative simplicity of those solutions and discussed some practical considerations.*

Introduction

Namit and Chen (1999) recently developed efficient and accurate algorithms for solving the <Q,r> inventory model with gamma lead-time demand, because "all of the methods for solving the ... model ... call for tabulated values and perhaps interpolation between them in every iteration". Although they have created an interesting and useful solution to the problem, Tyworth *et al.* (1996) have developed a method that does not require tabulated values to solve the problem (see also, Silver *et al.*, 1998, Keaton, 1995; Tyworth and Zeng, 1998). The purpose of this note is to demonstrate the relative simplicity of that approach and comment on the practical use of such models.

Proposed approach

Namit and Chen (1999) formulated the standard continuous review <Q,r> inventory model for the complete backorders case as follows:

$$K(Q,r) = A\frac{\lambda}{Q} + IC\left(\frac{Q}{2} + r - \mu\right) + \pi\frac{\lambda}{Q}\eta(r) \tag{1}$$

where K(Q,r) = expected total annual cost, Q = order quantity, r = reorder point, A = cost of placing an order, λ = expected annual demand, I = percentage of item cost per year that represents inventory carrying cost, C = cost per item, μ = expected lead-time demand, and π = backorder cost

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per unit backordered. The $\eta(r)$ element in equation (1) represents the expected backorders per replenishment cycle and is defined as:

$$\eta(r) = \int_r^\infty xh(x)dx - rH(r) \qquad (2)$$

where h(x) is the probability density function of lead-time demand (X) and

$$H(r) = \int_r^\infty h(x)dx,$$

which is the probability of a stock out. In addition, they assumed the distribution of X is gamma (α, β), with the mean expected demand during lead time (μ) = α/β. Although not stated explicitly by Namit and Chen, this model formulation rests on the assumption that the penalty cost per backorder (π) will be high enough to produce solutions for (Q^*, r^*) where safety stock ($r - \mu$) is non-negative.

The problem is to find the values of Q and r that jointly minimize expected total annual inventory system cost, K(Q,r). Namit and Chen

Model element	Symbol	Value units
Assumptions		
Expected annual demand	λ	55 per year
Cost per item	C	$50,000 per item
Cost of placing an order	A	$500 per order
Inventory carrying charge	I	0.1 per $ per year
Backorder cost	π	$2,000 per unit backordered
Lead-time demand parameter	α	3
Lead-time demand parameter	β	1
Derived elements		
Expected demand during lead time	μ	3 units
Expected backorders	$\eta(r)$	0.34 per replenishment cycle
Inventory policy variables		
Reorder point	r	4.05 units
Order quantity	Q	5.08 units
Solver solution procedure		
Set target cell (K) to minimum		
By changing	r, Q	
Subject to the constraints	$r \geq \mu$	
	$Q > 0$	
Results		
Ordering (A λ/Q)	OC	$5,414.53 per year
Inventory carrying [IC(Q/2 + r−μ)]	IC	$17,947.59 per year
Backorder [$\eta(r)\pi\lambda/Q$]	BC	$7,282.78 per year
Expected total annual cost	K	$30,644.90 per year

Table I. <Q,r> inventory model for gamma lead-time demand: an example for α = 3 and β =1 developed an iterative three-step procedure encompassing seven equations to solve the problem. When α gets large, however, certain computations in the procedure become tedious. They addressed this difficulty by first observing that the normal approximation of the gamma distribution is reasonable when $\alpha > 15$ and then creating a second algorithm based on the Hadley and Whiten (1963) iteration procedure for normal lead-time demand distributions. In addition, they presented an analytical solution for the special case when $\alpha = 1$, which is the exponential distribution.

Current approach

By contrast, Tyworth *et al.* (1996) developed a gamma loss function for $\eta(r)$, which enables the analyst to use non-linear optimization methods to solve the problem directly for any appropriate value of α. When the lead-time demand is gamma distributed, the expected units short (or backorders) per replenishment cycle can be defined as follows:

$$\eta(r) = \alpha\beta(1 - G_1(r)) - r(1 - G_0(r)) \tag{3}$$

Model element	Symbol	Value measure
Assumptions		
Expected annual demand	λ	55 per year
Cost per item	C	\$50,000 per item
Cost of placing an order	A	\$500 per order
Inventory carrying charge	I	0.1 per \$ per year
Backorder cost	π	\$2,000 per unit backordered
Lead-time demand parameter	α	16
Lead-time demand parameter	β	1
Derived elements		
Expected demand during lead time	μ	16 units
Expected backorders	$\eta(r)$	1.07 per replenishment cycle
Inventory policy variables		
Reorder point	r	17.29 units
Order quantity	Q	7.61 units
Solver solution procedure		
Set target cell (K) to minimum		
By changing	r, Q	
Subject to the constraints	$r \geq \mu$	
	$Q > 0$	
Results		
Ordering (A λ/Q)	OC	\$3,613.53 per year
Inventory carrying [IC(Q/2 + r $-\mu$)]	IC	\$25,459.27 per year
Backorder [η(r)$\pi\lambda$/Q]	BC	\$15,412.17 per year
Expected total annual cost	K	\$44,484.98 per year

Table II.
<Q,r> inventory model for gamma lead-time demand: an example for $\alpha = 16$ and $\beta = 1$

where G_1 = cumulative distribution function (cdf) of gamma ($\alpha + 1, \beta$) and G_0 = cdf of $G(\alpha, \beta)$. Given values for α and β, the problem can be formulated simply as follows:

Objective function: minimize $K(Q, r)$
Decision variables: Q, r
Constraints: $r \geq \mu$, $Q > 0$

The joint (Q^*, r^*) solution is solved easily and conveniently by using the non-linear optimization engines found in popular spreadsheet software. The solutions to the problem considered by Namit and Chen (1999), as well as the two special cases where $\alpha = 1$ and $\alpha = 16$, are illustrated in Tables I-III. Additionally, the Excel 97 file used to create the solutions may be downloaded from the following Web site: http://econqa.cba.uc.edu/~ganeshr/ijpdlm/ijpdlm.html

Practical considerations

Analysts must either assume or estimate the parameters of the gamma distribution (α and β). A practical way to accomplish this task is to gather

Model element	Symbol	Value units
Assumptions		
Expected annual demand	λ	55 per year
Cost per item	C	\$50,000 per item
Cost of placing an order	A	\$500 per order
Inventory carrying charge	I	0.1 per \$ per year
Backorder cost	π	\$2,000 per unit backordered
Lead-time demand parameter	α	1
Lead-time demand parameter	β	1
Derived elements		
Expected demand during lead time	μ	1 unit
Expected backorders	$\eta(r)$	0.20 per replenishment cycle
Inventory policy variables		
Reorder point	r	1.59 units
Order quantity	Q	4.46 units
Solvent solution procedure		
Set target cell (K) to minimum		
By changing	r, Q	
Subject to the constraints	$r \geq \mu$	
	$Q > 0$	
Results		
Ordering ($A\ \lambda/Q$)	OC	\$6,160.25 per year
Inventory carrying [$IC(Q/2 + r - \mu$]	IC	\$14,135.12 per year
Backorder [$\eta(r)\pi\lambda/Q$]	BC	\$5,000.00 per year
Expected total annual cost	K	\$25,295.38 per year

Table III. <Q,r> inventory model for gamma lead-time demand: an example for $\alpha = 1$ and $\beta = 1$

data on demands per day (D) and replenishment lead times (L). With these data in hand and assuming that demand (D) and lead time (L) are independent random variables, the analyst can then compute the mean (μ) and standard deviation (σ) of the lead-time demand distribution from estimates of the mean and standard deviation of demand (μ_D and σ_D) and lead time (μ_L and σ_L) with the following standard equations:

$$\mu = \mu_D \mu_L \tag{4}$$

$$\sigma = \sqrt{\mu_L \sigma_D^2 + \mu_D^2 \sigma_L^2} \tag{5}$$

Finally, the parameters of the gamma distribution are determined with μ and σ as follows: $\alpha = \sigma^2/\mu^2$ and $\beta = \sigma^2/\mu$.

Although estimates of the gamma parameters are easily developed by this method, practitioners may be more comfortable with the normal approximation of lead-time demand distribution. Recent research indicates that the normal approximation is robust for fast moving finished goods when the coefficient of variation (σ/μ) < 0.5 (Tyworth and O'Neil, 1996). Under these conditions, the practitioner might wish to skip the task of estimating the gamma parameters and simply use the unit normal loss function to define expected backorders per replenishment cycle, $\eta(r)$, as follows:

$$\eta(r) = [f(z) - z(1 - F(z))]\sigma \tag{6}$$

where f(z) = the probability density function of the unit normal distribution and F(z) is the cumulative density function and $z = (r - \mu)/\sigma$. Thus the practitioner only needs to know, or estimate, the values for μ and σ to solve the problem. The spreadsheet equivalent of equation (6) is = (NORMDIST(z, 0, 1, FALSE) – z*(1-NORMSDIST(z)))*σ.

Conclusion

This note demonstrated that a currently available approach for solving the <Q, r> inventory model for gamma lead-time demand does not require the use of tabulated values and is relatively simple to use. Unlike the Namit and Chen (1999) approach, which requires separate treatments for the special cases where $\alpha = 1$, $1 < \alpha \leq 15$, and $\alpha > 15$, the current approach easily handles each case with a comparatively simple structure. In addition, this note presented a practical method of estimating the parameters of the gamma distribution and described a convenient alternative formulation of the current model that assumes a normal approximation of lead-time demand, which is reasonable to use for fast moving finished goods and when $\mu/\sigma < 0.5$.

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